CCFU Proof 19

Explicit A_4: Companion Matrix from C_2

Given. $\operatorname{spec}(A_2) = \{\varphi,\ -1/\varphi\}$, where $\varphi = (1+\sqrt{5})/2$.

By Proof 17 (factor completion), the canonical four-mode spectrum is $\{\varphi,\ 1/\varphi,\ +1,\ -1\}$.

Step 1 — Characteristic polynomial.

$$\begin{aligned} p(\lambda) &= (\lambda-\varphi)(\lambda-1/\varphi)(\lambda-1)(\lambda+1) \\ &= (\lambda^2 - \sqrt{5}\,\lambda + 1)(\lambda^2-1) \\ &= \lambda^4 - \sqrt{5}\,\lambda^3 + \sqrt{5}\,\lambda - 1. \qquad \blacksquare \end{aligned}$$

Step 2 — Companion matrix.

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -\sqrt{5} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{5} \end{pmatrix}.$$

This is the companion matrix of $p(\lambda)$. By construction, $\operatorname{spec}(A_4) = \{\varphi, 1/\varphi, +1, -1\}$. $\blacksquare$

Step 3 — Invariant form. Let $D = \operatorname{diag}(\varphi, 1/\varphi, +1, -1)$ and define:

$$\tilde{G} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \operatorname{sig}(\tilde{G}) = (3,1).$$

Every nonzero entry $\tilde{G}_{ij}$ satisfies $\lambda_i\lambda_j = 1$, so $D^\top \tilde{G}\, D = \tilde{G}$. Let P be the eigenvector matrix ($A_4 = PDP^{-1}$, invertible since the four eigenvalues are distinct). Define $G_4 = P^{-\top}\tilde{G}\,P^{-1}$. Then $A_4^\top G_4\, A_4 = G_4$. $\blacksquare$

Step 4 — Determinant.

$$\det(A_4) = \varphi \cdot (1/\varphi) \cdot 1 \cdot (-1) = -1. \qquad \blacksquare$$

For later use. Since $A_4^\top G_4\, A_4 = G_4$, also $(A_4^2)^\top G_4\, A_4^2 = G_4$. Moreover $\det(A_4^2) = 1$ and $\operatorname{spec}(A_4^2) = \{\varphi^2, 1/\varphi^2, 1, 1\}$. Thus A_4^2 is a G_4-orthogonal, determinant-one map with positive hyperbolic spectrum. This is the boost component used in the translation-length proof (Proof 12).

Conclusion. A_4 is an explicit real 4×4 companion matrix, constructed from $\operatorname{spec}(A_2)$ alone via the factor completion of Proof 17. It preserves G_4 with $\operatorname{sig}(3,1)$ and has $\det = -1$. No free numerical input is introduced; the $\sqrt{5}$ scale is forced by $\Delta(C_2) = 5$.

This closes the external dependency of Proof 4 on Theory #15c.

[Dependency: Proof 17. No external references.]